# Inverse kinematic problem and boundary rigidity of Riemannian surfaces

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**Abstract.** Given a compact manifold with boundary with an unknown Riemannian metric. The problem is to reconstruct the metric in a class of conformal metrics from knowledge of lengths of all closed geodesics (kinematic data). An integral inequality is stated which implies uniqueness and stability for this problem. If the conformal class is not known a unique reconstruction is not possible since of shortage of information. It is proved that the list of all geodesic lengths is sufficient for unique determination of a Riemannian metric in a compact surface with boundary up to an automorphism which is identical on the boundary. Some related problems of integral geometry are studied.

**Key words:** Geodesic curve, Travel-time, Conjugate point, Geodesic flow, Hodograph, Geodesic integral transform.

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## 1 Introduction

Let $D$ be a compact domain in an Euclidean space $E^n$ with boundary $\partial D$ supplied with a conformal metric $\mathbf{g} = (\mathbf{n}\mathrm{d}s)^2$. Is it possible to recover the metric from boundary distance function $\tau(x, y)$, $x, y \in \partial D$? This question is known in geophysics as inverse kinematic or travel-time inversion problem; in this context $\mathbf{g} = (\mathbf{n}\mathrm{d}s)^2$, $\mathbf{n}$ is a refraction coefficient of a medium and boundary distance is called travel-time. In the pioneering papers of Herglotz [1], Wiechert [2] the problem was analytically solved for spherical Earth model under assumption that velocity $\mathbf{c} = 1/\mathbf{n}$ is a monotonically increasing function of depth. In the late seventies an important contribution was given by Mukhometov [8],[13], Mukhometov and Romanov [10], Bernstein and Gerver [12], Beylkin [15]. The uniqueness and stability of determination of a metric from travel-time were stated in the class of conformal Euclidean metrics. The arguments of [8],[13],[10],[12],[15] are based on the assumption that geodesics of both metrics are free of conjugate point (shortly f.c.p.). This assumption implies that for arbitrary points $x, y \in \partial D$ there is only one joining geodesic curve.

In a more general form the inverse kinematic problem is formulated as follows: given a compact manifold $D$ with boundary $\partial D$ and two conformal Riemannian metrics $\mathbf{g}_1$ and $\mathbf{g}_2$ in $D$ with equal boundary distance functions, do the metrics need to coincide? This problem was solved for the positive by Beylkin [15] and by Bernstein-Gerver [12] for arbitrary finsler metrics f.c.p. in domains $D$ in $\mathbb{R}^n$.

We address here the inverse kinematic problem for metrics in a more general setting. When conjugate points may appear the conformal coefficient can not be reconstructed from only boundary distance function. One need to know the *hodograph* of a metric which is a list of lengths of all closed geodesics in a manifold $D$. In Sec. 3 we prove the uniqueness and a stability estimate for the tensor $\mathbf{g}_1 - \mathbf{g}_2$ in terms of hodographs of these metrics. A necessary assumption is that

any geodesic ray in $D$ reaches the boundary transversely. Our estimate looks similar to that of [8],[13],[10],[12], [15] but is based on a different approach.

In Sec. 5 we consider the geodesic integral transform in a Riemannian manifold. We state a subelliptic estimate for a function in the manifold in terms of its transform without f.c.p. assumption. A standard elliptic 1/2-estimate is known only for f.c.p. geometries and this assumption is apparently necessary for ellipticity of the operator [17],[30].

In Sec. 6 we prove that a differential form in a Riemannian surface with zero integrals on all closed geodesics is exact. This fact was stated for f.c.p. metrics in [9],[23].

A more general geometrical problem was formulated by R. Michel [14] for a category of compact Riemannian manifolds $D$ with fixed boundary $\partial D$: given a metric $\mathbf{g}$ in $D$ and a diffeomorphism $\varphi : D \to D$ identical on $\partial D$, the pull back $\mathbf{g}' \doteq \varphi^* (\mathbf{g})$ is a (not conformal) Riemannian metric on $D$ with the same boundary distance function. The question is the inverse true: *for any two metrics* $\mathbf{g}$, $\mathbf{g}'$ *in* $D$ *with equal boundary distance functions* $\tau' = \tau$ *there exists such a isometry* $\varphi : (D, \mathbf{g}') \to (D, \mathbf{g})$ *identical on the boundary?* A manifold $(D, \mathbf{g})$ is called *boundary rigid* if for any metric $\mathbf{g}'$ (in a given class) with $\tau' = \tau$ there exists such an isometry $\varphi$. Michel [14] has shown that any compact connected domain admitting injective Riemannian immersion in a surface of constant curvature is boundary rigid in the class of f.c.p. metrics. Gromov [16] proved boundary rigidity for manifolds $D$ admitting a Riemannian immersion in a convex domain in a sphere or in an Euclidean space of arbitrary dimension. His arguments included Santalo's theorem [4]. Otal [20] extended Michel's theorem for arbitrary surfaces of strictly negative curvature (which are f.c.p.). For further results see Croke [21] and Sharafutdinov [22]. Stefanov and Uhlmann [25] and Eskin [24] proved boundary rigidity for metrics close to the Euclidean one, Sharafutdinov and Uhlmann [26] extended this result for Riemannian surfaces with no focal points (which are f.c.p.). Pestov and Uhlmann [28] have proved Michel's conjecture to the positive for a class of simple Riemannian surfaces. A manifold $(D, \mathbf{g})$ is called simple if it is simply connected, the boundary $\partial D$ is strictly convex and the metric has f.c.p. property. See also the survey [29].

The condition f.c.p. can not be omitted: it is easy to construct non equivalent metrics in a disc with the same distance function. On the other hand, conjugate points are inevitable for asymptotically Euclidean metrics. An exact result is: *any complete Riemannian f.c.p. metric on* $\mathbb{R}^2$ *which is isometric to the Euclidean metric outside a compact set must be isometric to the Euclidean metric* [3],[18].

We prove here that the condition f.c.p. can be omitted if we know the hodograph of a metric that is the list of lengths of all closed geodesics. In Sec. 7 and 8 we prove that the isometry class of a metric in a compact surface with boundary can be determined from knowledge of the hodograph. This fact implies that the hodograph is the only invariant of any isometry class of Riemannian surfaces with boundary. For a proof of the rigidity we follow the arguments of [28] using additional tools.

## 2 Preliminaries

**1.** Let $(D, \mathbf{g})$ be a compact Riemannian manifold of dimension $n$ with smooth boundary $\partial D$ that fulfils the conditions

(I) for any point $x \in D$ any geodesic curve $\gamma$ started at $x$ reaches the boundary $\partial D$ in both directions and

(II) the boundary $\partial D$ is strictly convex with respect to $\mathbf{g}$. This means that the second fundamental form of the boundary is positively definite at every point $p \in \partial D$. It follows that any geodesic curve must meet the boundary transversely.

We denote by $T(D)$ and $T^*(D)$ the tangent, respectively cotangent bundle on $D$. By means of a local coordinate system $x^1, ..., x^n$ we can write any tangent vector $\theta$ in the form $\theta = \theta^i \partial / \partial x^i$ and any

cotangent vector as $\xi = \xi_i \mathrm{d}x^i$. The scalar product for tangent vectors $\theta, \eta$ is $\langle \theta, \eta \rangle_x = g_{ij}(x)\,\theta^i \eta^j$ and for cotangent vectors $\alpha, \beta$ it is equal to $\langle \alpha, \beta \rangle_x = g^{ij}(x)\,\alpha_i \beta_j$. Let $S(D) = S_{\mathbf{g}}(D)$ be the bundle of unit vectors $\theta \in T(D)$ and $S^*_{\mathbf{g}}(D) \subset T^*(D)$ be the bundle of unit covectors. Notations $\partial S(D)$, $\partial S^*(D)$ mean the restrictions of these bundles to the boundary of $D$. For a unit vector $\theta$ the covector $\xi = \theta_*$ with coordinates $\xi_i = g_{ij}\theta^j, i = 1, ..., n$ also has unit norm; vice versa, for any $\xi \in S^*(D)$ the vector $\theta = \xi^*$, $\xi^i = g^{ij}\xi_j$ belongs to $S(D)$. The bundle $S^*(D)$ has a contact structure with the contact form $\sigma = \xi \mathrm{d}x \doteq \xi_i \mathrm{d}x^i$. The form $\mathrm{d}\sigma = \mathrm{d}\xi \mathrm{d}x = \mathrm{d}\xi_i \wedge \mathrm{d}x^i$ defines a symplectic structure in $T^*(D)$ that does not depend on a metric.

**2.** We call a vector $t \in T(S^*(D))$ *vertical* if $t(\pi^*(f)) = 0$ for any function $f$ defined in $D$ where $\pi : S^*(D) \to D$ is a natural map. A *horizontal* vector is an element of the bundle $T_h(S^*(D)) \doteq S^*(D) \times_D T(D)$. There is a canonical map $\pi_h : T(S^*(D)) \to T_h(S^*(D))$ generated by natural maps $\pi_T, \mathrm{d}\pi$ in the diagram

$$\begin{array}{ccccc} & & T(S^*(D)) & & \\ & \pi_T \swarrow & \pi_h \downarrow & \searrow \mathrm{d}\pi & \\ S^*(D) & \rightarrow & S^*(D) \times_D T(D) & \leftarrow & T(D) \end{array}$$

For an arbitrary $t \in T(S^*(D))$ the image $\pi_h(t)$ is called the horizontal part of $t$. There is an exact sequence of bundles over $S^*(D)$ :

$$0 \to T_v(S^*(D)) \to T(S^*(D)) \to T_h(S^*(D)) \to 0$$

where $T_v(S^*(D))$ is the bundle of vertical vectors.

**3.** Let $(\Omega^*, \mathrm{d})$ be the complex of smooth differential forms in $T^*(D)$. The complex $\Omega^*_S$ of differential forms on $S^*(D)$ is by definition the restriction to $S^*(D)$ of the complex $\Omega^*/J$ where $J$ is the ideal in the exterior algebra $\Omega^*$ generated by the form $\mathrm{d}\mathbf{g}(x, \xi)$. The

$$\mathrm{d}_\xi f = \sum \frac{\partial f}{\partial \xi_i} \mathrm{d}\xi_i$$

is well defined in $\Omega^*$ which does not depend on the coordinate system in $D$ and $\mathrm{d}_\xi \mathrm{d} = -\mathrm{d}\mathrm{d}_\xi$.

Let $v = v_1 \wedge ... \wedge v_l$ be a tangent multivector at a point $(x, \xi) \in S^*(D)$. We denote by symbol $v \rhd \alpha$ the contraction of a differential form $\alpha \in \Omega^k$ by a $l$-multivector $v$, $l \leq k$ a differential form $\beta$ of degree $k - l$ defined as follows

$$\beta \doteq (v_l \wedge ... \wedge v_1) \rhd \alpha = v_l \rhd (...v_2 \rhd (v_1 \rhd \alpha))$$
$$(w \rhd \alpha)(t_1, ..., t_{k-1}) = \sum_j (-1)^j \alpha(t_1, ..., t_j, w, t_{j+1}, ..., t_{k-1})$$

**4** Let $\gamma(x, \theta)$ be a full geodesic ray starting at $x$ in the direction of a unit vector $\theta$, Let $y = y(x, \theta) \in \partial D$ be the arrival point of $\gamma$ and $\zeta = \zeta(x, \theta)$ is the outward unit tangent vector to $\gamma$ at $y$. The map

$$T_{\mathbf{g}} : S(D) \to \partial S(D),\ (x, \theta) \mapsto (y, \zeta)$$

is smooth due to conditions (I,II). We can write this map in terms of unit cotangent vectors $T_{\mathbf{g}} : S^*(D) \to \partial S^*(D)$, $(x, \xi) \mapsto (y, \eta)$ where $\xi = \theta_*, \eta = \zeta_*$. We call it *travel map* of the metric $\mathbf{g}$. The length of a full geodesic $\gamma(x, \theta)$ is the number

$$\tau_{\mathbf{g}}(x, \xi) = \int_{\gamma(x,\theta)} \mathrm{d}_{\mathbf{g}} s$$

We call to the function $\tau_{\mathbf{g}}$ defined on $S^*(D)$ *travel-time* function.

**Proposition 1** *For any point* $(x, \xi) \in S^* (D)$ *we have*

$$\mathrm{d}\tau_{\mathbf{g}} (x, \xi) = -\xi \mathrm{d}x + \eta \mathrm{d}y \tag{1}$$

*where* $(y, \eta) = T_{\mathbf{g}} (x, \xi)$ *.*

**Proof.** We can write

$$\mathrm{d}\tau_{\mathbf{g}} (x, \xi) = \left. \mathrm{d}\tau_{\mathbf{g}} (x, \xi)\right|_{\mathrm{d}x=0} + \left. \mathrm{d}\tau_{\mathbf{g}} (x, \xi)\right|_{\mathrm{d}y=0}$$

The first term is equal to $\eta \mathrm{d}y$ since the front of the wave propagating from $x$ is orthogonal to the unit forward covector $\eta$. By the same reason the second term is equal to $-\xi \mathrm{d}x$ since the unit covector $\xi$ is directed backwards to the wave propagating from $y$. ►

We can write (1) in a more simple form $\mathrm{d}\tau_{\mathbf{g}} = -\sigma + T^*_{\mathbf{g}} \sigma$.

# 3 Stability of the inverse kinematic problem

**Theorem 2** *For arbitrary conformal metrics* $\mathbf{g}$ *and* $\tilde{\mathbf{g}} = \mathbf{r}^2 \mathbf{g}$, $\mathbf{r} = \mathbf{r}(x) > 0$ *in* $D$ *satisfying (I,II) the equation holds*

$$\left|\mathrm{S}^{n-1}\right| \int_D (\omega - 1)^2 \mathrm{d}_{\mathbf{g}} V \leq \nu_n \int_{\partial S^*(D)} \mathrm{d}(\tilde{\tau} - \tau) \wedge \mathrm{d}_\xi (\tilde{\tau} - \tau) \wedge (\mathrm{d}\xi \mathrm{d}x)^{\wedge n-2} \tag{2}$$

*where* $\tilde{\tau}$ *is the travel-time function of the metric* $\tilde{\mathbf{g}}$, $\mathrm{d}_{\mathbf{g}} V$ *is the volume density defined by* $\mathbf{g}$, $\left|\mathrm{S}^{n-1}\right|$ *is the area of the unit sphere and* $\nu_n = (-1)^{n(n-1)/2} (n-1)!$.

**Proof.** The function $\tilde{\tau}\left(x, \tilde{\xi}\right)$ was defined on the manifold $S^*_{\tilde{\mathbf{g}}} (D)$ of covectors $\tilde{\xi}$ of unit $\tilde{\mathbf{g}}$-norm. For an arbitrary covector $\xi \in S^*_{\mathbf{g}} (D)$ we have $\tilde{\mathbf{g}} (x, \xi) = \mathbf{r}^{-2} (x) \mathbf{g} (x, \xi) = \mathbf{r}^{-2} (x)$. The covector $\tilde{\xi} = \mathbf{r}(x) \xi$ fulfils $\tilde{\mathbf{g}}\left(x, \tilde{\xi}\right) = 1$ and lies in $S^*_{\mathbf{g}} (D)$ .Thus the function $\tilde{\tau}(x, \mathbf{r}(x) \xi)$ is well defined on $S^*(D)$ as well as the difference $\rho(x, \xi) = \tilde{\tau}(x, \mathbf{r}(x)\xi) - \tau(x, \xi)$. Consider the forms

$$\Lambda = \mathrm{d}\rho \wedge \mathrm{d}_\xi \rho \wedge \Sigma,\ \ \Sigma = (\mathrm{d}\xi \mathrm{d}x)^{\wedge n-2}$$

and have

$$\mathrm{d}\Lambda = \mathrm{d}\rho \wedge \mathrm{d}_\xi \mathrm{d}\rho \wedge \Sigma$$

By (1) $\mathrm{d}\tilde{\tau} = -\tilde{\xi}\mathrm{d}x + \tilde{\eta}\mathrm{d}\tilde{y}$, where $\tilde{\xi} \in S^*_{\tilde{\mathbf{g}}} (D)$ is an arbitrary covector of length

$$\mathrm{d}\rho = -\xi \mathrm{d}x + \eta \mathrm{d}y + \mathbf{r}\xi \mathrm{d}x - \tilde{\eta}\mathrm{d}\tilde{y}$$

where $(\tilde{y}, \tilde{\eta}) = T_{\tilde{\mathbf{g}}} (x, \mathbf{r}(x)\xi)$ . Direct calculate gives a sum of 16 terms

$$\begin{aligned} \mathrm{d}\rho \wedge \mathrm{d}_\xi \mathrm{d}\rho = & (\xi \mathrm{d}x - \eta \mathrm{d}y) \wedge (\mathrm{d}\xi \mathrm{d}x - \mathrm{d}_\xi (\eta \mathrm{d}y)) \\ & + (\mathbf{r}\xi \mathrm{d}x - \tilde{\eta}\mathrm{d}\tilde{y}) \wedge (\mathbf{r}\mathrm{d}\xi \mathrm{d}x - \mathrm{d}_\xi (\tilde{\eta}\mathrm{d}\tilde{y})) \\ & - (\mathbf{r}\xi \mathrm{d}x - \tilde{\eta}\mathrm{d}\tilde{y}) \wedge (\mathrm{d}\xi \mathrm{d}x - \mathrm{d}_\xi (\eta \mathrm{d}y)) \\ & - (\xi \mathrm{d}x - \eta \mathrm{d}y) \wedge (\mathbf{r}\mathrm{d}\xi \mathrm{d}x - \mathrm{d}_\xi (\tilde{\eta}\mathrm{d}\tilde{y})) \end{aligned} \tag{3}$$

We write the first line of (3) in the form $(A + B) \wedge (C + D)$ and find

$$\begin{aligned} A \wedge C \wedge \Sigma &= \xi \mathrm{d}x \wedge \mathrm{d}\xi \mathrm{d}x \wedge \Sigma = \xi \mathrm{d}x \wedge (\mathrm{d}\xi \mathrm{d}x)^{n-1} \\ &= (-1)^{n(n-\mathbf{1})/2} \sum \xi_{i_1} \mathrm{d}\xi_{i_2} \wedge \ldots \wedge \mathrm{d}\xi_{i_n} \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_n} \end{aligned}$$

where the sum is taken over all permutations $(i_1, ..., i_n)$ of $(1, ..., n)$. The result equals

$$A \wedge C \wedge \Sigma = \nu_n \mathrm{d}\omega \wedge \mathrm{d}V,\ \nu_n = (-1)^{n(n-1)/2}(n-1)!,$$

where $\mathrm{d}\omega = \sum (-1)^{i-1} \mathrm{d}\omega_1 \wedge ...\widehat{\mathrm{d}\omega_i}... \wedge \mathrm{d}\omega_n$ is the area form on the Euclidean unit sphere $\mathrm{S}^{n-1}$ and

$$\mathrm{d}V = G^{1/2}\mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^n,\ G = \det\{g_{ij}\} \tag{4}$$

**Lemma 3** *The field* $\lambda = (\theta, \partial_x) - (h_x, \partial_\xi)$, $h(x,\xi) = \sqrt{\mathbf{g}(x,\xi)}, h_x = \xi^* = \theta$ *is a generator of the geodesic flow in* $S^*(D)$. *It fulfils the equation* $\mathrm{d}y(\lambda) = 0$.

**Proof.** The field $\lambda$ generates a flow in $T^*(D)$. It is the geodesic flow since it preserves the function $h^2 = \mathbf{g}$. It preserves also the travel map $T_{\mathbf{g}}$ and in particular the function $y = y(x,\xi)$. We have $h_\xi = \mathbf{g}_\xi/2h = \theta/h = \theta$ since $h = 1$ in $S^*(D)$. ▶

Fix an arbitrary point $(x, \xi^0) \in S^*(D)$ and choose horizontal vectors $\theta_1, ..., \theta_{n-1}$ at this point such that the frame $\theta_0, \theta_1, ..., \theta_{n-1} \in T_x(D)$ is an orthogonal basic where $\theta_0 = (\xi^0)^*$. Let $\xi^i,\ i = 0, ..., n-1$ be an orthogonal basis in $T_x^*(D)$ such that $(\xi^i, \theta_j) = \delta_j^i,\ i,j = 0, ..., n-1$. Consider the rotation group action $R_k : \mathrm{S}^1 \to S_x(D)$ defined for $k = 1, ..., n-1$ and arbitrary point $\theta = \sum a^i \theta_i$ by

$$R_k(\varphi)\theta = (\cos\varphi a^0 - \sin\varphi a^k)\theta_0 + (\sin\varphi a^0 + \cos\varphi a^k)\theta_k + \sum_{j\neq 0,k} a^j\theta_j,\ a^j = (\theta, \xi^j)$$

A generator is the vertical field

$$\varepsilon_k = -a^k\frac{\partial}{\partial a^0} + a^0\frac{\partial}{\partial a^k} \tag{5}$$

These fields fulfil the equations

$$\varepsilon_k(\theta_0) = \varepsilon_k(\theta)|_{\theta=\theta_0} = \theta_k,\ \varepsilon_k(\theta_k) = -\theta_0 \tag{6}$$

and can be defined in $S_x^*$ by $\varepsilon_k(\xi) = \varepsilon_k(\xi^*)$, $\xi^* \in S(U)$ that is

$$\varepsilon_k(\xi) = -(\xi, \theta_k)\xi^0 + (\xi, \theta_0)\xi^k \tag{7}$$

According to Lemma 3 $\theta_0$ is the horizontal part of $\lambda$. The fields $\lambda_k = [\varepsilon_k, \lambda]$, $k = 1, ..., n-1$ are tangent to $S^*(D)$; by (6) the horizontal part of $\lambda_k$ is equal to $\theta_k$. The multivector $\varepsilon_1 \wedge ... \wedge \varepsilon_{n-1} \wedge \lambda \wedge \lambda_1 \wedge ... \wedge \lambda_{n-1}$ fulfils the equation

$$(\lambda_{n-1} \wedge ... \wedge \lambda_1 \wedge \lambda \wedge \varepsilon_{n-1} \wedge ... \wedge \varepsilon_1) \rhd \mathrm{d}\omega \wedge \mathrm{d}V = 1 \tag{8}$$

at the point $(x, \xi^0)$. The equations

$$\lambda \rhd \mathrm{d}\xi\mathrm{d}x = \mathrm{d}h = 0,\ k = 1, ..., n-1 \tag{9}$$

are satisfied in $\Omega_S^*$ since $\mathrm{d}h = 1/2\mathrm{d}\mathbf{g}$ in $S^*(D)$. It follows that

$$B \wedge C \wedge \Sigma = \eta\mathrm{d}y \wedge (\mathrm{d}\xi\mathrm{d}x)^{\wedge n-1} = 0 \tag{10}$$

since each factor vanishes after contraction by $\lambda$. For the first factor it follows from Lemma 3 and this is true for the second factor by (9). By (7) for any $k, j = 1, ..., n-1$

$$(\lambda_j \wedge \varepsilon_k) \rhd \mathrm{d}\xi\mathrm{d}x = \mathrm{d}\xi(\varepsilon_k)\mathrm{d}x(\theta_j) = \left(-(\xi, \theta_k)\xi^0 + (\xi, \theta_0)\xi^k, \theta_j\right)\big|_{\xi=\xi_0} = \delta_j^k$$

This implies for $k=1,...,n-1$

$$\left(\lambda_{n-1}\wedge...\widehat{\lambda}_k...\wedge\lambda_1\wedge\varepsilon_{n-1}\wedge...\widehat{\varepsilon}_k...\wedge\varepsilon_1\right)\rhd\Sigma=(-1)^{(n-3)(n-2)/2}(n-2)! \tag{11}$$

Further we have

$$\begin{aligned}A\wedge D-B\wedge C&=\xi\mathrm{d}x\wedge\mathrm{d}_\xi(\eta\mathrm{d}y)-\eta\mathrm{d}y\wedge\mathrm{d}\xi\mathrm{d}x=-\mathrm{d}_\xi(\xi\mathrm{d}x\wedge\eta\mathrm{d}y)\\(A\wedge D-B\wedge C)\wedge\Sigma&=-\mathrm{d}_\xi(\xi\mathrm{d}x\wedge\eta\mathrm{d}y\wedge\Sigma)\end{aligned}$$

since $\mathrm{d}_\xi\Sigma=0$. By the Kelvin-Stokes theorem this yields

$$\int_{S^*(D)}A\wedge D\wedge\Sigma-\int_{S^*(D)}B\wedge C\wedge\Sigma=0$$

By (10) both integrals in the left hand side vanish. Calculate the last term in the first line of (3):

$$B\wedge D\wedge\Sigma=\eta\mathrm{d}y\wedge\mathrm{d}_\xi\eta\mathrm{d}y\wedge\Sigma+\eta\mathrm{d}y\wedge\eta\mathrm{d}_\xi\mathrm{d}y\wedge\Sigma$$

The first term equals zero since each factor annihilates by contraction with $\lambda$ since of Lemma 3 and (9): To evaluate the second term we note that all the factors except for the second one are contracted to zero by $\lambda$. We have for any $k=1,...,n-1$

$$(\lambda_k\wedge\lambda\wedge\varepsilon_k)\rhd B\wedge D=(\lambda_k\wedge\lambda\wedge\varepsilon_k)\rhd\eta\mathrm{d}y\wedge\eta\mathrm{d}_\xi\mathrm{d}y=\eta\mathrm{d}y(\lambda_k)(\lambda\wedge\varepsilon_k)\rhd\eta\mathrm{d}_\xi\mathrm{d}y$$

since $\mathrm{d}y(\lambda)=0$. To calculate the second factor we use the formula

$$(v\wedge\varepsilon)\rhd\mathrm{d}_\xi a=(L_\varepsilon a)(v)=L_\varepsilon(a(v))-a([\varepsilon,v])=\varepsilon(a(v))-a(L_\varepsilon v)$$

where $a$ is a 1-form, $v,\varepsilon$ are tangent fields and $L$ means a Lie derivative. Take $a=\mathrm{d}y$ and $v=\lambda$ and obtain $(\lambda\wedge\varepsilon_k)\rhd\mathrm{d}_\xi\mathrm{d}y=\varepsilon_k(\mathrm{d}y(\lambda))-\mathrm{d}y(\lambda_k)=-\mathrm{d}y(\lambda_k)$ which yields

$$(\lambda_k\wedge\lambda\wedge\varepsilon_k)\rhd B\wedge D=-(\eta\mathrm{d}y(\lambda_k))^2 \tag{12}$$

Applying (11),(12) for $k=1,...,n-1$ and taking in account (8) to obtain the equation

$$B\wedge D\wedge\Sigma=\frac{\nu_n}{n-1}\sum(\eta\mathrm{d}y(\lambda_k))^2\,\mathrm{d}\omega\wedge\mathrm{d}V$$

since $(n-3)(n-2)+2=n(n-1)\mod(4)$. The integral of all terms in the first line of (3) equals

$$\int_{S^*(D)}(A+B)\wedge(C+D)\wedge\Sigma=\nu_n\int_{S^*(D)}\left(1+\frac{1}{n-1}\sum(\eta\mathrm{d}y(\lambda_k))^2\right)\mathrm{d}\omega\wedge\mathrm{d}V$$

We denote the second line in (3) by $\left(\tilde{A}+\tilde{B}\right)\wedge\left(\tilde{C}+\tilde{D}\right)$ and obtain by similar calculations

$$\int\left(\tilde{A}+\tilde{B}\right)\wedge\left(\tilde{C}+\tilde{D}\right)\wedge\Sigma=\nu_n\int\left(\mathbf{r}^2+\frac{1}{n-1}\sum\left(\tilde{\eta}\mathrm{d}\tilde{y}\left(\tilde{\lambda}_k\right)\right)^2\right)\mathrm{d}\omega\wedge\mathrm{d}V$$

where $\tilde{\lambda}_k=\left[\varepsilon_k,\tilde{\lambda}\right],\ k=1,...,n-1$ and $\tilde{\lambda}=\left(\mathbf{r}^{-1}\theta,\partial_x\right)-\left(\left(\mathbf{r}^{-1}h\right)_x,\partial_\xi\right)$ is the generator of the geodesic flow of the metric $\tilde{\mathbf{g}}$ in the bundle $S^*_{\tilde{\mathbf{g}}}(D)$ and . The third and the forth lines are equal to $-\left(\tilde{A}+\tilde{B}\right)\wedge(C+D)$ and $-(A+B)\wedge\left(\tilde{C}+\tilde{D}\right)$ respectively. Integrating we get

$$\begin{aligned}&-\int\left[\tilde{A}\wedge C+\tilde{B}\wedge D+A\wedge\tilde{C}+B\wedge\tilde{D}\right]\wedge\Sigma\\&=-\nu_n\int 2\mathbf{r}+\frac{2}{n-1}\sum\left[\eta\mathrm{d}y(\lambda_k)\ \ \tilde{\eta}\mathrm{d}\tilde{y}\left(\tilde{\lambda}_k\right)\right]\mathrm{d}\omega\wedge\mathrm{d}V\end{aligned}$$

but

$$\int\left[\tilde{A}\wedge D-B\wedge\tilde{C}\right]\wedge\Sigma=\mathbf{r}\xi\mathrm{d}x\wedge\mathrm{d}_{\xi}\left(\eta\mathrm{d}y\right)-\eta\mathrm{d}y\wedge\mathbf{r}\mathrm{d}\xi\mathrm{d}x=-\int\mathrm{d}_{\xi}\left(\mathbf{r}\xi\mathrm{d}x\wedge\eta\mathrm{d}y\wedge\Sigma\right)=0 \quad (13)$$

$$\int\left[A\wedge\tilde{D}-\tilde{B}\wedge C\right]\wedge\Sigma=0$$

The integral of the form

$$\tilde{A}\wedge D\wedge\Sigma=\eta\mathrm{d}y\wedge\mathbf{r}\mathrm{d}\xi\mathrm{d}x\wedge\Sigma$$

vanish since each factor of the integrand is contracted to zero by the field $\lambda$. The same is true for the form $A\wedge\tilde{D}\wedge\Sigma$ if we contract by $\tilde{\lambda}$. By (13) the same is true for the forms $B\wedge\tilde{C}\wedge\Sigma$ and $\tilde{B}\wedge C\wedge\Sigma.'$ The sum of all integrals results to

$$\frac{1}{\nu_n\left|\mathrm{S}^{n-1}\right|}\int_{S^*(D)}\mathrm{d}\Lambda=\int_{S^*(D)}\left(\mathbf{r}-1\right)^2\mathrm{d}V+\frac{1}{n-1}\int_{S^*(D)}\sum_{k=1}^{n-1}\left[\tilde{\eta}\mathrm{d}\tilde{y}\left(\tilde{\lambda}_k\right)-\eta\mathrm{d}y\left(\lambda_k\right)\right]^2\mathrm{d}\omega\wedge\mathrm{d}V \quad (14)$$

The Kelvin-Stokes formula yields

$$\int_{S^*(D)}\mathrm{d}\Lambda=\int_{\partial S^*(D)}\Lambda$$

which together with (14) implies inequality (2) if we omit the positive term. ►

**Corollary 4** *Conformal equivalent metrics coincide if* $\mathbf{r}\left(p\right)=1$ *and travel-time functions are equal on the boundary that is* $\tilde{\tau}\left(p,\xi\right)\equiv\tau\left(p,\xi\right)$ *for* $\left(p,\xi\right)\in\partial S_{\mathbf{g}}^*\left(D\right).$

**Remark.** The conformal coefficient $\mathbf{r}\left(p\right)$ can be found comparing boundary distances $\tau\left(p,q\right)$ and $\tilde{\tau}\left(p,q\right)$ for close points $p,q\in\partial D$ :

$$\mathbf{r}\left(p\right)=\lim_{q\to p}\frac{\tilde{\tau}\left(p,q\right)}{\tau\left(p,q\right)}$$

**Remark.** The condition (I) is violated if there is a wave-guide in $D$ which causes infinite geodesics or closed geodesic that do not appear on the boundary. Then Corollary 4 fails since travel-time data are not complete. The ambiguity of reconstruction of the velocity field is well studied in the case where a refraction coefficient in a ball depends only on depth [5],[6].

# 4 Interpretation of the second term

The omitted positive term in (14) can be interpreted in more invariant way. For a point $\left(x,\xi_0\right)\in S^*\left(D\right)$ we consider the functional

$$\Theta_{\mathbf{g}}:T_{\xi}\left(S_x^*\right)\to\mathbb{R},\ \varepsilon_k\mapsto\eta\left(x,\xi\right)\mathrm{d}y\left(\lambda_k\right)=T_{\mathbf{g}}^*\sigma\left(\lambda_k\right)\left(x,\xi\right),\ k=1,...,n-1$$

defined on tangent vectors $\varepsilon_k\in T_{\xi}\left(S_x^*\right)$ as in (7) and extended linearly to the whole space is the space $T_{\xi}\left(S_x^*\right)$. For any point $\xi\in S_x^*$ the tangent space $T_{\xi}\left(S_x^*\right)$ is generated by the vectors $\varepsilon_1,...,\varepsilon_{n-1}$ as in (5). Therefore the functional $\Theta_{\mathbf{g}}$ is well defined as a linear map

$$\Theta_{\mathbf{g}}\left(x,\xi\right):T_{\xi}\left(S_x^*\right)\to\mathbb{R}\ ,\ \sum_{k=1}^{n-1}c_k\varepsilon_k\mapsto\sum c_kT_{\mathbf{g}}^*\sigma\left(\lambda_k\right),\ c_k\in\mathbb{R}$$

The functional $\Theta_{\tilde{\mathbf{g}}}\left(x,\tilde{\xi}\right):\ \varepsilon_k\mapsto T^*_{\tilde{\mathbf{g}}}\tilde{\lambda}_k$ is defined in the same way by means of the vectors $\tilde{\theta}_k=\mathbf{r}^{-1}(x)\,\theta_k$ , $k=0,1,...,n-1$ for the argument $\tilde{\xi}=\mathbf{r}(x)\,\xi$. The difference

$$\Theta_{\tilde{\mathbf{g}}}\left(x,\tilde{\xi}\right)-\Theta_{\mathbf{g}}(x,\xi):T_\xi\left(S^*_x\right)\to\mathbb{R}$$

is well defined. The vertical vectors $\varepsilon_1,...,\varepsilon_{n-1}\in T_\omega\left(S^*_x\right)$ form an orthogonal frame for any which implies

$$\sum\left[\tilde{\eta}\mathrm{d}\tilde{y}\left(\tilde{\lambda}_k(x,\xi)\right)-\eta\mathrm{d}y\left(\lambda_k(x,\xi)\right)\right]^2=\left|\Theta_{\tilde{\mathbf{g}}}\left(x,\tilde{\xi}\right)-\Theta_{\mathbf{g}}(x,\xi)\right|^2$$

The right hand side does not depend on choice of fields $\theta_1,...,\theta_{n-1}$ or on choice of a local coordinate system in $D$. Finally we can write the positive term in (14) in the invariant form

$$\int_{S^*(D)}\sum\left[\tilde{\eta}\mathrm{d}\tilde{y}\left(\tilde{\lambda}_k\right)-\eta\mathrm{d}y\left(\lambda_k\right)\right]^2\mathrm{d}\omega\wedge\mathrm{d}V=\int_{S^*(D)}\left|\Theta_{\tilde{\mathbf{g}}}\left(x,\tilde{\xi}\right)-\Theta_{\mathbf{g}}(x,\xi)\right|^2\mathrm{d}\omega\wedge\mathrm{d}V$$

# 5 Subelliptic estimate for geodesic integral transform

Given a Riemannian manifold $(D,\mathbf{g})$ we define geodesic integral transform of a function $f$ in $D$ by means of the integral

$$If(\gamma)=\int_\gamma f(p)\,\mathrm{d}_{\mathbf{g}}s$$

defined for the family of closed geodesic curves $\gamma$.

**Theorem 5** *For arbitrary metric* $\mathbf{g}$ *in a compact manifold* $D$ *with boundary that satisfies (I,II) and any real function* $f\in L_2(D)$ *the equation holds*

$$\left|\mathrm{S}^{n-1}\right|\int_D f^2\mathrm{d}V\le\nu_n\int_{\partial S^*(D)}\mathrm{d}F\wedge\mathrm{d}_\xi F\wedge(\mathrm{d}\sigma)^{\wedge n-2}\tag{15}$$

*where* $F=If$.

**Proof.** Substitute $\mathbf{r}=1+\varepsilon f$ in (2) and get

$$\rho(x,\theta)=-\varepsilon F(\gamma(x,\theta))+O\left(\varepsilon^2\right)$$

$$\varepsilon^2(n-1)\left|\mathrm{S}^{n-1}\right|\int_D f^2\mathrm{d}V\le\varepsilon^2(n-1)\,\nu_n\int_{\partial S^*(D)}\mathrm{d}F\wedge\mathrm{d}_\xi F\wedge(\mathrm{d}\sigma)^{\wedge n-2}+O\left(\varepsilon^2\right)$$

Taking the limit as $\varepsilon\to0$ and cancelling the factor $n-1$ we obtain (15). ▶

**Remark.** A similar estimate for the integral of $f\mathrm{d}V$ is due to Mukhometov [7] ($n=2$) and Romanov [11] ($n\ge2$) when conjugate points are absent.

**Corollary 6** *The inequality holds*

$$\|f\|^0_D\le\frac{(n-1)!}{\left|\mathrm{S}^{n-1}\right|}\|\nabla If\|^0_{\partial S^*(D)}\le C\,\|If\|^1_{\partial S^*(D)}\tag{16}$$

*in terms of Sobolev's* $L_2$*-norms.*

For a family $\Phi$ of curves satisfying (**i**) without conjugate points a better estimate for functions $f$ with compact support is known:

$$\|f\|^{\alpha} \leq C_{\alpha,\beta} \|If\|^{\alpha+1/2} + C_{\beta} \|f\|^{\beta} \tag{17}$$

Here $\alpha$ and $\beta$ are arbitrary real and the Sobolev class of order $\alpha + 1/2$ is the best possible, see e.g. [30] for the case of surface $D$. This fact follows from ellipticity of the operator $I^*I$ [17]. Therefore (16) looks as a subelliptic estimate (in the sense of FIO theory) with 1/2 loss against the elliptic case. It does not depend on the condition f.c.p.. Meantime we conclude from (16) that if $\alpha \geq 1/2$ the term $\|f\|^{\beta}$ can be omitted in (17).

# 6 Differential forms with vanishing geodesic integrals

**Theorem 7** *Let $(D, \mathbf{g})$ be a compact Riemannian surface with boundary satisfying conditions (I,II). Let $\alpha$ be a 1-differential form of the class $C^1(D)$ such that*

$$\int_{\gamma} \alpha = 0 \tag{18}$$

*for any closed geodesic curve $\gamma$ in $D$. There exists a function $f \in C^0(D)$ vanishing on $\partial D$ such that $\alpha = \mathrm{d}f$ in $D$.*

The inverse is of course true since of condition (I). This statement was proved for surfaces [9] and for manifolds f.c.p. of arbitrary dimension [19],[22], [23],[26].

**Proof.** We use the notations of Sec.2. The function

$$A(x,\xi) = \int_{\gamma(x,\theta)} \alpha, \ \xi = \theta_* \tag{19}$$

belongs to $C^2(S^*(D))$ and vanishes for any $x \in \partial D$ and arbitrary $\theta$ since of (18). Consider the form $\beta(x,\xi) = \alpha(x) - \mathrm{d}A(x,\xi)$.

**Lemma 8** *The equation $\beta(\lambda) = 0$ holds at any point $(x,\xi) \in S^*(D)$.*

**Proof.** Let $y = y(t)$ be a parametrization of the geodesic $\gamma(x,\theta)$ such that $y(0) = x$ and $|y'| = 1$. Denote $\xi(t) = y'(t)^*$ and we have

$$\frac{\mathrm{d}}{\mathrm{d}t} A(y(t), \xi(t))|_{t=0} = \left(\theta, \frac{\partial A}{\partial x}\right) - \left(h_x, \frac{\partial A}{\partial \xi}\right) = \mathrm{d}A(\lambda)|_{(x,\xi)}$$

since $\gamma$ is a trajectory of the hamiltonian function $h$. By (19) the left hand side is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma(y(t),\theta(t))} \alpha \Bigg|_{t=0} = \mathrm{d}\alpha(\theta) = \alpha(\lambda)$$

which yields $\beta(\lambda) = \alpha(\lambda) - \mathrm{d}A(\lambda) = 0$ and completes the proof. ►

By Proposition 12 surface $D$ possesses an orientation and a coordinate covering by oriented conformal maps in $\mathbb{C}$. For each coordinate domain $U$ the metric tensor has a form $\mathbf{g}(x,\xi) = r^2(x)|\xi|^2$ where $|\xi|^2 = (\xi^0)^2 + (\xi^1)^2$ and $x^1, x^2$ are standard coordinates in $\mathbb{C}$ and $r(x) > 0$. This implies that the bundle $S^*(D)$ is a rotation surface $\{(x,\xi); r(x)|\xi| = 1\}$ in $T^*(D)$. Coordinate mappings preserve orientation. Therefore the vertical field $\varepsilon = \partial/\partial\varphi = \xi^0\partial/\partial\xi^1 - \xi^1\partial/\partial\xi^0$ is globally defined in the bundle $S^*(D)$. The commutator $\mu \doteq [\varepsilon, \lambda]$ is equal to $(*\theta, \partial_x) + (*h_x, \partial_\xi)$ where $*v$ means rotation of a vector $v$ by $\pi/2$, that is $*v = (-v_1, v_0)$ for a $v = (v_0, v_1)$ and $\theta = r^2\xi$.

**Lemma 9** *We have $\beta(\mu) = 0$.*

**Proof.** The product $B = \beta \wedge \mathrm{d}_\xi \beta$ is a volume form in $S^*(D)$. We calculate its integral in two ways. By (**??**) we have

$$(\mu \wedge \lambda \wedge \varepsilon) \rhd \beta \wedge \mathrm{d}_\xi \beta = \beta(\mu)\left[\varepsilon(\beta(\lambda)) - \beta(\mu)\right] = -\beta(\mu)^2 \tag{20}$$

since $\beta(\lambda) \equiv 0$. The fields $\mu, \lambda, \varepsilon$ form a frame in $S^*(D)$ and $|\varepsilon \wedge \lambda \wedge \mu| = |\theta|^2 = r^2(x) = G^{-1/2}$. This yields

$$\int_{S^*(D)} B = -\int \beta(\mu)^2 \,\mathrm{d}\omega \wedge \mathrm{d}V \tag{21}$$

On the other hand

$$B = -(\alpha - \mathrm{d}A) \wedge \mathrm{d}_\xi \mathrm{d}A = \mathrm{d}_\xi(\alpha \wedge \mathrm{d}A) + \mathrm{d}(A \wedge \mathrm{d}_\xi \mathrm{d}A)$$

since $\mathrm{d}_\xi \alpha = 0$. Therefore by the Kelvin-Stokes theorem

$$\int_{S^*(D)} B = \int \mathrm{d}_\xi(\alpha \wedge \mathrm{d}A) + \int \mathrm{d}(A \wedge \mathrm{d}_\xi \mathrm{d}A) = \int_{\partial S^*(D)} A \wedge \mathrm{d}_\xi \mathrm{d}A = 0$$

since $A$ vanishes on $\partial D$. Comparing with (21) we complete the proof. ▶

**Lemma 10** *The system of fields $\lambda, \mu, \varepsilon$ is involutive and*

$$[\lambda, \mu] = r(*\xi, r_x)\lambda + r(\xi, r_x)\mu + \left(r\Delta r + |\nabla r|^2\right)\varepsilon \tag{22}$$

**Proof.** By a direct calculation one can check the equation $[\varepsilon, \mu] = -\lambda$ and (22). ▶

**Lemma 11** *We have $\mathrm{d}\beta(\lambda, \mu) = \rho\varepsilon(A)$ where $\rho = r\Delta r + |\nabla r|^2$.*

**Proof.** For an arbitrary 1-form $\omega$ and smooth fields $s, t$

$$\mathrm{d}\omega(s, t) = s(\omega(t)) - t(\omega(s)) - \omega([s, t])$$

Apply this equation to $\beta$ :

$$\mathrm{d}\beta(\lambda, \mu) = \lambda(\beta(\mu)) - \mu(\beta(\lambda)) - \beta([\lambda, \mu])$$

By Lemmas 9,8 and (22) the first and the second terms vanish and the third term is equal to $\beta([\lambda, \mu]) = \rho\varepsilon(A)$. ▶

Consider the form $b = A \wedge \mathrm{d}\beta$ and have $\mathrm{d}_\xi b = \mathrm{d}_\xi A \wedge \mathrm{d}\beta$ since $\mathrm{d}_\xi \mathrm{d}\beta = \mathrm{d}_\xi \mathrm{d}\alpha = 0$. By Lemma 11 we have

$$\mathrm{d}b(\varepsilon, \lambda, \mu) = \mathrm{d}_\xi A(\varepsilon)\,\mathrm{d}\beta(\lambda, \mu) = \rho(\varepsilon(A))^2$$

Integrating along a fibre $S_x^*$ we obtain

$$0 = \int_{S_x^*(D)} \mathrm{d}_\xi b = \rho(x) \int_{S_x^*} \varepsilon(A)^2 \,\mathrm{d}\varphi$$

It follows that $\rho\varepsilon(A) = 0$ everywhere in $D$. By Lemma 11 the function $\mathrm{d}\beta(\lambda, \mu) = \mathrm{d}\alpha(\theta, *\theta)$ vanishes in $D$. This implies that $\mathrm{d}\alpha = 0$. Take a point $q \in \partial D$ and define a function

$$f(x) = \int_{\delta(q,x)} \alpha$$

in $D$ where $\delta(q,x)$ is an arbitrary simple $C^1$-curve connecting $q$ and $x$. The integral does not depend on the curve since $\alpha$ is closed and $D$ is simply connected by Proposition 12. The function $f$ is smooth in $D$ up to the boundary and $\mathrm{d}f = \alpha$. Check that $f = 0$ on the boundary. Take an arbitrary point $p \in \partial D$; there exists a geodesic $\gamma(q,p)$ connecting $q$ and $p$ and

$$f(p) = \int_{\gamma(q,p)} \alpha = 0$$

by (18). This completes the proof of Theorem. ►

**Proposition 12** *Any surface $D$ with boundary that fulfils (I,II) is orientable and simply connected.*

**Proof.** The boundary $\partial D$ is a union of several circles $C_1, ..., C_n$. We stick some discs $D_1, ..., D_n$ to $D$ along these circles so that the amalgam $\tilde{D} = D \sqcup (D_1 \cup ... \cup D_n)$ is a smooth compact surface without boundary. We can extend the metric $\mathbf{g}$ to a smooth metric $\tilde{\mathbf{g}}$ in $\tilde{D}$. If $\tilde{D}$ is not orientable, then the group $H_1\left(\tilde{D}, \mathbb{Z}_2\right)$ is not trivial. Let $h$ be a non zero homology class. Suppose that there exists a shortest curve $\gamma \in h$ that is contained in $D \setminus \partial D$. It is a geodesic curve in $D$ that does not touch the boundary. This is impossible since of (I). Otherwise we look for a shortest curve $\gamma_1 \in h$ that is contained in $D$. This curve can not be a union of circles $C_i$ since the homology class of each circle $C_i$ in $\tilde{D}$ is trivial. Therefore the chain $\gamma_1 \setminus \partial D$ is a non empty union of geodesics tangent to $\partial D$ in its end points. This contradicts (II) since no nontrivial geodesic can be tangent to the boundary. This implies that $D$ is orientable.

Check that it is simply connected. If it is not the case, the group $H_1(D, \mathbb{Z})$ is not trivial. Then the above arguments lead to a contradiction with (I) or (II). ►

# 7 Travel map from hodograph

Let $\mathbf{g}$ be a metric in a manifold $D$ with a boundary $\partial D$ that fulfils conditions (I,II) of Sec. 2. Consider the map

$$H_{\mathbf{g}} \doteq \pi \times \pi t_{\mathbf{g}} \times \tau_{\mathbf{g}} : \partial_+ S(D) \to \partial D \times \partial D \times \mathbb{R}$$

where $\pi : \partial S(D) \to \partial D$ is the natural projection and $\tau_{\mathbf{g}}(p,\theta)$ is the length of the geodesic $\gamma(p,\theta)$ and $t_{\mathbf{g}}$ is the restriction of the travel map $T_{\mathbf{g}}$ to the variety $\partial_+ S(D)$. By $\partial_\pm S(D)$ we denote the set of pairs $(p,\theta) \in \partial S(D)$ such that $\pm(\nu,\theta) \geq 0$. We call the image of $H_{\mathbf{g}}$ *hodograph* of the metric $\mathbf{g}$. We assume that

(III) the set of geodesics $\gamma = \gamma(p,q)$ in $D$ that have caustic points at both points $p$ and $q$ is nowhere dense in the variety of all closed geodesics. We shall show that this property can be checked from the hodograph and prove hodograph rigidity of a class of Riemannian metrics on compact surfaces:

**Theorem 13** *Any compact Riemannian surface $(D, \mathbf{g})$ with boundary satisfying (I,II,III) is uniquely determined by its hodograph up to an isometry of $D$ identical on the boundary.*

Consider the map $h_{\mathbf{g}} = \pi \times \pi t_{\mathbf{g}} : \partial_+ S(D) \to \partial D \times \partial D$, $h_{\mathbf{g}}(p,\theta) = (p,q)$ where $q$ is the end of the geodesic $\gamma(p,\theta)$. This is a smooth map of manifolds of dimension $2n-2$.

**Proposition 14** *If for some $p_0 \in \partial D$ the family of geodesics $\gamma(p_0,\theta)$ has a caustic point as $\theta = \theta_0$ then $(p_0,\theta_0)$ is a critical point of $h_{\mathbf{g}}$ and vice versa.*

**Proof.** The condition $\det \partial q(p_0, \theta_0)/\partial\theta = 0$ indicates that the family of geodesics $\gamma(p_0, \theta)$ has caustic at a point $q_0$ as $\theta = \theta_0$. The Jacobian matrix of $h_{\mathbf{g}}$ is

$$J = \begin{pmatrix} \frac{\partial q}{\partial p} & \frac{\partial q}{\partial \theta} \\ I & 0 \end{pmatrix}$$

and $\det J = \pm \det \partial q/\partial\theta$. Vanishing of $\det \partial q/\partial\theta$ is equivalent to the equation $\det J = 0$ which means that the geodesic $\gamma(p_0, \theta_0)$ is a critical point of $h_{\mathbf{g}}$. ►

Each geodesic $\gamma(p_0, \theta_0)$ appears in $\partial_+ S(D)$ once again as $\gamma(q_0, -\zeta_0)$ where $(q_0, \zeta_0) = t_{\mathbf{g}}(p_0, \theta_0)$. If $q_0$ is a caustic point of the family $\gamma(p_0, \theta)$ at $\theta = \theta_0$ then $\det J = 0$ hence $p_0$ is a caustic point of the family $\gamma(q_0 - \zeta)$ at $\zeta = \zeta_0$. Therefore condition (III) can be formulated as follows: the set $K$ of critical points of $h_{\mathbf{g}}$ is nowhere dense.

**Theorem 15** *For any metric* $\mathbf{g}$ *satisfying (I,II,III) the map* $t_{\mathbf{g}}$ *uniquely determined from knowledge of the hodograph* $\Gamma_{\mathbf{g}}$.

**Proof.** Let $\delta_0 \in G_{p_0} \cap G_{q_0}$ be an arbitrary noncritical point of $h_{\mathbf{g}}$. There exists a neighborhood $V \subset \partial D \times \partial D$ of $(p_0, q_0)$ and a smooth family of geodesics $\gamma = \delta(p, q)$ defined for $(p, q) \in V$ such that $\delta(p_0, q_0) = \delta_0$. Travel-time $\tau(\delta(p, q))$ is a smooth function in $V$ and by (1) we have

$$\mathrm{d}\tau(\delta(p, q)) = -\xi(p, q)\,\mathrm{d}p + \eta(p, q)\,\mathrm{d}q \tag{23}$$

where $\tau$ denotes length of a geodesic. By means of (23) one can determine for each geodesic $\delta(p, q)$ the restriction $\xi'$ of the initial covector $\xi(p, q) \in \partial S^*(D)$ to $T_p(\partial D)$ and the restriction $\eta'$ of the exit covector $\eta(p, q) \in \partial S^*(D)$ to the tangent plane at $q$. It is sufficient for determination of both unit vectors if we know the tensor $\mathbf{g}$ on the boundary. For arbitrary metrics $\mathbf{g}_1, \mathbf{g}_2$ in $D$ with the same hodograph there exists a smooth automorphism $\psi$ of $D$ identical on the boundary such that $\psi^*(\mathbf{g}_2)$ coincides with $\mathbf{g}_1$ on the boundary. This follows from [14] Proposition 2.4 which does not depend on the assumption f.c.p. since only arbitrarily short boundary distances are used. Therefore we can assume that two metrics $\mathbf{g}_1, \mathbf{g}_2$ in $D$ which have the same hodograph coincide at any point $p \in \partial D$. Now to determine a covector $\xi$ we write $\xi = \xi' + s\nu$ where $\nu$ is the inward unit conormal to $\partial D$ and have $\mathbf{g}(p, \xi) = \mathbf{g}(p, \xi' + s\nu) = \mathbf{g}(p, \xi') + s\mathbf{g}(p, \xi', \nu) + s^2 = 1$ where $\mathbf{g} = \mathbf{g}_1 = \mathbf{g}_2$. This is a quadratic equation with two real roots $s_{1,2}$ such that $s_1 + s_2 = -2\mathbf{g}(p, \xi', \nu)$. We look for a root such that $\nu(\theta) < 0$ where $2\theta = \mathbf{g}_\xi(p, \xi)$ (the case $\nu(\theta) = 0$ is trivial since $\tau(p, \theta) = 0$). We have

$$2\nu(\theta) = (\mathbf{g}_\xi(p, \xi'), \nu) + s(\mathbf{g}_\xi(p, \nu), \nu) = 2\mathbf{g}(p, \xi', \nu) + 2s = 2s - s_1 - s_2$$

There is two options $s = s_1$ or $s = s_2$ in this equation which yield $2\nu(\theta) = \pm(s_1 - s_2)$. Only one choice provides a unique solution $\xi = \xi' + s\nu$ of the inequality $\nu(\theta) < 0$ which means that $\theta$ is outward vector. This implies that the initial covector $\xi = \xi(p, q)$ is the same for the metrics $\mathbf{g}_1$ and $\mathbf{g}_2$. The same is true for the final covector $\eta = \eta(p, q)$ for which $\nu(\zeta) > 0$. We have $t_{\mathbf{g}}(p, \xi) = (q, \eta)$ for the geodesics $\delta(p, q)$ of both metrics. Next, we determine vectors $\theta(p, q) = \xi^*(p, q)$ and $\zeta(p, q) = \eta^*(q, \eta)$ by means of the known metric tensor on the boundary and reconstruct the travel map by $t_{\mathbf{g}}(p, \theta(p, q)) = (p, \zeta(p, q))$ for any pair $(p, q) \in V$.

**Lemma 16** *If a point* $(p_0, q_0) \in \partial D \times \partial D$ *is a noncritical value of* $h_{\mathbf{g}}$, *there is only finite number of geodesics joining* $p_0$ *and* $q_0$.

**Proof of Lemma.** Consider the set of tangent vectors $\Theta$ at $p_0$ such that $q(p_0, \theta) = q_0$, $\theta \in \Theta$. This set is closed and has no accumulation point since otherwise $(p_0, q_0)$ is a critical of $h_{\mathbf{g}}$. The set $\Theta$ is finite since the manifold $\partial_+ S(D) \cap \pi^{-1}(p_0)$ is compact. ►

Thus one only need to recognize the graphs $\mathrm{G}_k, k = 1, ..., \omega$ of smooth functions $\tau_k(p,q)$ defined in a neighborhood of $(p_0, q_0)$ such that the finite union $\cup \mathrm{G}_k$ coincides with the set $\Gamma_{\mathbf{g}}$ in a neighborhood of the line $L_0 = \{(p_0, q_0) \times \mathbb{R}\}$. By (23) each graph $\mathrm{G}_k$ has an affine approximation $\tau_k(p_0, q_0) + \mathrm{d}\tau_k(p_0, q_0)$, $k = 1, ..., \omega$. and restrictions of these linear functions to the tangent space $T_p(\partial D) \times T_q(\partial D)$ are all different. It can be uniquely done. Now we know all the smooth functions $\tau_k(p,q) = \tau(\delta_k(p,q))$, $k = 1, ..., K$ defined in a neighborhood $V$ of $(p_0, q_0)$.

Vice versa, suppose that the hodograph $\Gamma_{\mathbf{g}}$ can be represented in a neighborhood of a line $L_0$ as a union of a finite number of graphs of $C^1$-functions $\tau_1, ..., \tau_\omega$ defined in a neighborhood $V$ of the point $(p_0, q_0)$ such that

$$\mathrm{d}\tau_k(p,q) = -\xi_k(p,q)\,\mathrm{d}p + \eta_k(p,q)\,\mathrm{d}q$$

with some continuous covectors $\xi_k, \eta_k$ such that all the triples $(\tau_k, \xi_k, \eta_k)$, $k = 1, ..., \omega$ are different. The set $C_{\mathbf{g}}$ of critical values of $h_{\mathbf{g}}$ is closed (since $h_{\mathbf{g}}$ is proper) and has zero measure by Sard's theorem. Therefore for any point $(\tilde{p}, \tilde{q}) \in V \setminus C_{\mathbf{g}}$ there exists a neighborhood $V'$ of noncritical points and each function $\tau_k$ coincides with a function $\tau(\delta(\tilde{p}, \tilde{q}))$ as above. The vectors $\xi_k, \eta_k$ are projections to the boundary of initial and final tangent vectors to the geodesic $\delta(\tilde{p}, \tilde{q})$ and we have $t_{\mathbf{g}}(p, \xi_k) = (q, \eta_k)$ in $V'$. This equation holds by continuity for any pair of points $(p,q) \in V$.

In this way we have determined the travel map in the set $\partial S(D) \setminus Z$, where $Z = h_{\mathbf{g}}^{-1}(C_{\mathbf{g}})$.

**Lemma 17** *The set $Z$ is closed and nowhere dense.*

This Lemma implies that the travel map is uniquely determined by continuity on the whole manifold $\partial S(D)$. This completes the proof of Theorem 15. ►

**Proof of Lemma 17.** The set $K$ of critical points of $h_{\mathbf{g}}$ is nowhere dense by (III) and its complement $R$ is open. The set of critical values $C_{\mathbf{g}}$ has zero measure by Sard's theorem. The set $Z = h_{\mathbf{g}}^{-1}(C)$ is closed and the intersection $R \cap Z$ also has zero measure. The union $(R \cap Z) \cup K$ is also nowhere dense and coincides with $Z$. ►

# 8 Hodograph rigidity

Now we complete the proof of Theorem 13. Let $\mathbf{g}$ be a metric in a compact surface $D$ with boundary. Consider the Laplace-Beltrami operator $\Delta_{\mathbf{g}}$ on $D$ and the Dirichlet to Neumann operator $\Lambda_{\mathbf{g}}$ defined by $\Lambda_{\mathbf{g}}(h_0) = \mathrm{d}f(\nu)$ where $\nu$ is the unit inward normal field on $\partial D$ and

$$\Delta_{\mathbf{g}} f = 0 \text{ on } D,\ f = h_0 \text{ on } \partial D$$

The Laplace-Beltrami operator of a Riemannian surface is conformal covariant that is $\mathbf{s}\Delta_{\mathbf{sg}} = \Delta_{\mathbf{g}}$ for any positive function $\mathbf{s}$ in $D$ which implies $\Lambda_{\mathbf{sg}} = \Lambda_{\mathbf{g}}$.

**Theorem 18** *Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be Riemannian metrics in a compact surface with boundary that fulfil conditions (I,II). Then the equation $t_{\mathbf{g}_1} = t_{\mathbf{g}_2}$ implies $\Lambda_{\mathbf{g}_1} = \Lambda_{\mathbf{g}_2}$.*

**Proof.** One can repeat the proof of [28] Theorem 1.3 with few changes that are necessary to extend the arguments for metrics with conjugate points.

**Lemma 19** *For an arbitrary $g \in H^0(D)$ the equation $I^* w = g$ has a solution $w \in H^{-1}(\partial_+ S(D))$ where $H^k(M)$ means a Sobolev class of functions on a compact manifold $M$.*

**Proof**. The ray transform

$$I : f \mapsto If(p,\theta) = \int_{\gamma(p,\theta)} f \mathrm{d}_{\mathbf{g}} s$$

can be extended to a bounded operator $I : L_2(D) \to L_2(\partial_+ S(D))$. The adjoint operator $I^* : L_2(\partial_+ S(D)) \to L_2(D)$ is given by

$$I^* u(x) = \int_{S_x(D)} u\left(T_{\mathbf{g}}^-(x,\theta)\right) \mathrm{d}\omega$$

where the map $T_{\mathbf{g}}^- : S(D) \to \partial_+ S(D)$ is defined by $T_{\mathbf{g}}^-(x,\theta) = (q, -\zeta)$ where $(q,\zeta) = T_{\mathbf{g}}(x, -\theta)$ and $\mathrm{d}\omega$ is the angular measure in a circle. This definition has a meaning for arbitrary distribution $u$ in $\partial_+ S(D)$ since $T_- : S(D) \to \partial_+ S(D)$ is a proper surjection. By Corollary 6 we have

$$\int_D |f|^2 \mathrm{d}V \leq C \int_{\partial_+ S(D)} |\nabla(If)|^2 \mathrm{d}\theta \wedge \mathrm{d}s$$

Consider the family of Sobolev spaces $H^k(D)$ and $H^k(\partial S(D))$, $k \in \mathbb{R}$; we denote norms in both scales $\|\cdot\|^k$. The above inequality implies

$$\|f\|_D^0 \leq C \|If\|_{\partial_+ S(D)}^1 \tag{24}$$

for Sobolev's norms. We define a functional $u$ on the image of the operator $I : H^1(D) \to H^1(\partial_+ S(D))$ by $\langle u, If \rangle = \langle g, f \rangle$. We also set $\langle u, v \rangle = 0$ for any function $v \in H^1(\partial S(D))$ supported in $\partial_- S(D)$. By (24) we have

$$|u(If)| = |g(f)| \leq \|g\|^0 \|f\|^0 \leq C \|g\|^0 \|If\|^1$$

hence $u$ is continuous with respect to the norm $\|\cdot\|^1$ on $\partial_+ S(D)$. We extend $u$ to a functional $w$ on the space $H^1(\partial S(D))$ by means of F. Riesz's theorem. The functional $u$ can be identified with an element of the space $H_2^{-1}(\partial S(D))$. It is a distribution in $\partial S(D)$ supported in $\partial_+ S(D)$. The identity $\langle w, If \rangle = \langle g, f \rangle$ holds at least for arbitrary $f \in H^1(D)$. This yields $g = I^* w$ that is

$$g(x) = \int u\left(T_{\mathbf{g}}^-(x,\theta)\right) \mathrm{d}\omega$$

where the right hand side is defined in distribution sense. ▶

Next arguments of [28] can be applied to any distribution-solution $w$ in the same way as to a smooth solution. One more point in [28] that need to be completed is the fact used in Theorem 1.6: *given a Riemannian metric* $\mathbf{g}$ *in* $D$ *that fulfils (I,II) and a smooth 1-differential form* $\alpha$ *in* $D$ *such that* $\int_\gamma \alpha = 0$ *for any geodesic curve* $\gamma$, *there exists a function* $f$ *in* $D$ *such that* $\alpha = \mathrm{d}f$. This fact is contained in Theorem 7 that does not depend on the condition f.c.p. This completes the proof of Theorem 18. ▶

**Proof of Theorem 13.** Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be metrics as in Theorem 18. By Theorem 15 we have $t_{\mathbf{g}_1} = t_{\mathbf{g}_2}$ and by Theorem 18 $\Lambda_{\mathbf{g}_1} = \Lambda_{\mathbf{g}_2}$.A general result of [27] now implies that there exists an boundary trivial diffeomorphism $\varphi$ and a smooth positive function $\mathbf{s}$ in $D$ such that $\mathbf{s} = 1$ on $\partial D$ and $\mathbf{g}_1 = \mathbf{s}\mathbf{g}_3$ where $\mathbf{g}_3 = \varphi^*(\mathbf{g}_2)$. We have $t_{\mathbf{g}_2} = t_{\mathbf{g}_3}$ since the hodographs of $\mathbf{g}_2$ and $\mathbf{g}_3$ are the same. Therefore the conformal metrics $\mathbf{g}_1$ and $\mathbf{g}_3$ have equal travel maps $t_{\mathbf{g}_1} = t_{\mathbf{g}_3}$. By Theorem 2 this equation implies $\mathbf{s} = 1$. It follows that $\mathbf{g}_1 = \mathbf{g}_3$ that is $\mathbf{g}_1 = \varphi^*(\mathbf{g}_2)$ which completes the proof of Theorem 13. ▶

# 9 Volume from hodograph

**Theorem 20** *Let* $(D, \mathbf{g})$ *be a compact Riemannian manifold of dimension* $n$ *with boundary satisfying (I,II). The volume* $V_{\mathbf{g}}(D)$ *is expressed in terms of the travel-time function as follows*

$$V_{\mathbf{g}}(D) = -\frac{1}{\nu_n |\mathrm{S}^{n-1}|} \int_{\partial S_{\mathbf{g}}^*(D)} \tau_{\mathbf{g}}(x,\xi) (\mathrm{d}\xi \mathrm{d}x)^{\wedge n-1} \tag{25}$$

**Remark.** Mukhometov [13] gave a representation for the volume of a f.c.p. metric in terms of an integral over $\partial D \times \partial D$. In the case $n = 2$ it coincides with Santaló's formula [4].

**Proof.** The form $\upsilon = \tau\,(\mathrm{d}\xi\mathrm{d}x)^{\wedge n-1}$ is well defined on $S^*(D)$ and by (1) we have

$$\mathrm{d}\upsilon = \mathrm{d}\tau \wedge (\mathrm{d}\xi\mathrm{d}x)^{\wedge n-1} = -\xi\mathrm{d}x \wedge (\mathrm{d}\xi\mathrm{d}x)^{\wedge n-1} + \eta\mathrm{d}y \wedge (\mathrm{d}\xi\mathrm{d}x)^{\wedge n-1}$$

The second term vanishes at any point $x \in D$ since the contraction by the geodesic field $\lambda$ kills both factors $\eta\mathrm{d}y$ and $\mathrm{d}\xi\mathrm{d}x$ (see Sec. 3). The first term equals

$$\xi\mathrm{d}x \wedge (\mathrm{d}\xi\mathrm{d}x)^{\wedge n-1} = \nu_n \mathrm{d}\omega \wedge \mathrm{d}_{\mathbf{g}}V$$

where $\mathrm{d}\omega$ is the canonical area form on $\mathrm{S}^{n-1}$. Integrating we obtain

$$\int_{\partial S^*(D)} \upsilon = \int_{S^*(D)} \mathrm{d}\upsilon = -\int \xi\mathrm{d}x \wedge (\mathrm{d}\xi\mathrm{d}x)^{\wedge n-1} = \nu_n \int_{S^*(D)} \mathrm{d}\omega \wedge \mathrm{d}_{\mathbf{g}}V = \nu_n \left|\mathrm{S}^{n-1}\right| V_{\mathbf{g}}(D)$$

and (25) follows. ►

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